Well-posedness of a nonlinear integro-differential problem and its rearranged formulation [*][†]

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Abstract

We study the existence and uniqueness of solutions of a nonlinear integro-differential problem which we reformulate introducing the notion of the decreasing rearrangement of the solution. A dimensional reduction of the problem is obtained and a detailed analysis of the properties of the solutions of the model is provided. Finally, a fast numerical method is devised and implemented to show the performance of the model when typical image processing tasks such as filtering and segmentation are performed.

Keywords: Integro-differential equation, existence, uniqueness, neighborhood filters, decreasing rearrangement, denoising, segmentation.

1   Introduction

This article is devoted to the study of the nonlinear integro-differential problem

$$\partial_t u(t,\mathbf{x}) = \int_\Omega \mathcal{K}_h(u(t,\mathbf{y}) - u(t,\mathbf{x}))(u(t,\mathbf{y}) - u(t,\mathbf{x}))d\mathbf{y} \tag{1}$$
$$+ \lambda(u_0(\mathbf{x}) - u(t,\mathbf{x})),$$
$$u(0,\mathbf{x}) = u_0(\mathbf{x}) \tag{2}$$

for $(t,\mathbf{x}) \in Q_T = (0,T) \times \Omega$. Here, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ denotes an open and bounded set, $T > 0$, $\lambda > 0$ and $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$. The *range* kernel $\mathcal{K}_h$ is given as a rescaling $\mathcal{K}_h(\xi) = \mathcal{K}(\xi/h)$ of a kernel $\mathcal{K}$ satisfying the usual properties of nonnegativity and smoothness. We shall give the precise assumptions in Section 3. We shall refer to problem (1)-(2) as to problem $P(\Omega, u_0)$. The main results contained in this article are:

- Theorem 1. The well-posedness of problem $P(\Omega, u_0)$, the stability property of its solutions with respect to the initial datum, and the time invariance of the level set structure of its solutions.

- Theorem 2. The equivalence between solutions of problem $P(\Omega, u_0)$ and the one-dimensional problem $P(\Omega_*, u_{0*})$, where $\Omega_* = (0, |\Omega|)$, and u_{0*} is the decreasing rearrangement of u_0, see Section 2 for definitions.

- Theorem 3. The asymptotic behavior of the solution of problem $P(\Omega_*, u_{0*})$ with respect to the window size parameter, h, as a shock filter.

Problem $P(\Omega, u_0)$ is related to some problems arising in Image Analysis, Population Dynamics and other disciplines. The general formulation in (1) includes, for example, a time-continuous version of the Neighborhood filter (NF) operator:

$$\mathrm{NF}^h\, u(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_\Omega e^{-\frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{h^2}} u(\mathbf{y}) d\mathbf{y},$$

where h is a positive constant, and $C(\mathbf{x}) = \int_\Omega \exp\left(-|u(\mathbf{x}) - u(\mathbf{y})|^2 h^{-2}\right) d\mathbf{y}$ is a normalization factor. In terms of the notation introduced for problem $P(\Omega, u_0)$ the NF is recovered setting $\mathcal{K}(s) = \exp(-s^2)$ and $\lambda = 0$. This well known denoising filter is usually employed in the image community through an iterative scheme,

$$u^{(n+1)}(\mathbf{x}) = \frac{1}{C_n(\mathbf{x})} \int_\Omega \mathcal{K}_h(u^{(n)}(\mathbf{x}) - u^{(n)}(\mathbf{y})) u^{(n)}(\mathbf{y}) d\mathbf{y}, \tag{3}$$

with $C_n(\mathbf{x}) = \int_\Omega \mathcal{K}_h(u^{(n)}(\mathbf{x}) - u^{(n)}(\mathbf{y})) d\mathbf{y}$. It is the simplest particular case of other related filters involving nonlocal terms, notably the Yaroslavsky filter [31, 32], the Bilateral filter [27, 29], and the Nonlocal Means filter [8].

These methods have been introduced in the last decades as efficient alternatives to local methods such as those expressed in terms of nonlinear diffusion partial differential equations (PDE's), among which the pioneering nonlinear anti-diffusive model of Perona and Malik [19], the theoretical approach of Álvarez et al. [1] and the celebrated ROF model of Rudin et al. [22]. We refer the reader to [9] for a review comparing these local and non-local methods.

Another image processing task encapsulated by problem $P(\Omega, u_0)$ is the *histogram prescription*, used for image contrast enhancement: Given an initial image u_0, find a companion image u such that u and u_0 share the same level sets structure, and the histogram distribution of u is given by a prescribed function Ψ. A widely used choice is $\Psi(s) = s$, implying that u has a uniform histogram distribution. In this case, $\mathcal{K}(s) = sign^-(s)/s$ and λ is related to the image size and its dynamic range, see Sapiro and Caselles [23] for the formulation and analysis of the problem. Nonlinear integro-differential of the form

$$\partial_t u(t, \mathbf{x}) = \int_\Omega (u(t, \mathbf{y}) - u(t, \mathbf{x})) w(\mathbf{x} - \mathbf{y}) d\mathbf{y} \tag{4}$$

and other nonlinear variations of it have also been recently used (Andreu et al. [6]) to model diffusion processes in Population Dynamics and other areas. More precisely, if $u(t, \mathbf{x})$ is thought of as a density at the point $\mathbf{x}$ at time t and $w(\mathbf{x} - \mathbf{y})$ is thought of as the probability distribution of jumping from location $\mathbf{y}$ to location $\mathbf{x}$, then $\int_\Omega u(t, \mathbf{y}) w(\mathbf{x} - \mathbf{y}) d\mathbf{y}$ is the rate at which individuals are arriving at position $\mathbf{x}$ from all other places and $-u(t, \mathbf{x}) = -\int_\Omega u(t, \mathbf{x}) w(\mathbf{x} - \mathbf{y}) d\mathbf{y}$ is the rate at which they are leaving location $\mathbf{x}$. In the absence of external or internal sources this consideration leads immediately to the fact that the density u satisfies the equation (4).

These kind of equations are called nonlocal diffusion equations since in them the diffusion of the density u at a point $\mathbf{x}$ and time t depends not only on $u(t, \mathbf{x})$ but also on the values of u in a set determined (and weighted) by the space kernel w. A thoroughfull study of this problem may be found in the monograph by Andreu et al. [6]. Observe that in problem $P(\Omega, u_0)$, the

space kernel is taken as $w \equiv 1$, meaning that the influence of nonlocal diffusion is spread to the whole domain.

As noticed by Sapiro and Caselles [23] for the histogram prescription problem, and later by Kindermann et al. [16] for the iterative Neighborhood filter (3), or by Andreu et al. [6] for continuous time problems like (4), these formulations may be deduced from variational considerations. For instance, in [16], the authors consider, for $u \in L^2(\Omega)$, the functional

$$J(u) = \int_{\Omega \times \Omega} g(u(\mathbf{x}) - u(\mathbf{y}))w(\mathbf{x} - \mathbf{y})d\mathbf{x}d\mathbf{y}, \tag{5}$$

with an appropriate spatial kernel w, and a differentiable filter function g. Then, the authors formally deduce the equation for the critical points of J. These critical points coincide with the fixed points of the nonlocal filters they study. For instance, if $g(s) = \int_0^s \mathcal{K}_h(\sqrt{\sigma})d\sigma$ and $w \equiv 1$, the critical points satisfy

$$u(\mathbf{x}) = \frac{1}{C(\mathbf{x})} \int_\Omega \mathcal{K}_h(u(\mathbf{x}) - u(\mathbf{y}))u(\mathbf{y})d\mathbf{y},$$

which can be solved through a fixed point iteration mimicking the iterative Neighborhood filter scheme (3). On the other hand, choosing $g(s) = s$ (or some suitable nonlinear variant) and considering a gradient descent method to approximate the stationary solution, equation (4) is deduced. Similarly, $g(s) = |s|$ and $w \equiv 1$ leads to the histogram prescription problem.

Although the functional (5) is not convex in general, Kindermann et al. prove that when $\mathcal{K}$ is the Gaussian kernel then the addition to J of a convex fidelity term, e.g.

$$\tilde{J}(u; u_0) = J(u) + \lambda \|u - u_0\|^2_{L^2(\Omega)},$$

gives, for $\lambda > 0$ large enough, a convex functional $\tilde{J}$, see [16, Theorem 3.1].

Thus, the functional $\tilde{J}$ may be seen as the starting point for the deduction of problem $P(\Omega, u_0)$, representing the continuous gradient descent formulation of the minimization problem modeling Gaussian image denoising. Notice that although the convexity of $\tilde{J}$ is only ensured for λ large enough, the results obtained in this article are independent of such value, and only the usual non-negativity condition on λ is assumed.

The outline of the article is as follows. In Section 2, we introduce some basic notation and the definition of *decreasing rearrangement* of a function. This is later used to show the equivalence between the general problem $P(\Omega, u_0)$ and the reformulation $P(\Omega_*, u_{0*})$ in terms of a problem with a identical structure but defined in a one-dimensional space domain. This technique was already used in [12] for dealing with the time-discrete version of problem $P(\Omega, u_0)$, in the form of the iterative scheme (3). See also [13, 14] for the problem with non-uniform spatial kernel. In Section 3, we state our main results. Then, in Section 4, we introduce a discretization scheme for the efficient approximation of solutions of problem $P(\Omega, u_0)$, and demonstrate its performance with some examples. In Section 5, we provide the proofs of our results, and finally, in Section 6, we give our conclusions.

2 The decreasing rearrangement

Given an open and bounded (measurable) set $\Omega \subset \mathbb{R}^d$, $(d \geq 1)$ let us denote by $|\Omega|$ its Lebesgue measure and set $\Omega_* = (0, |\Omega|)$. For a Lebesgue measurable function $u : \Omega \to \mathbb{R}$, the function

$q \in \mathbb{R} \to m_u(q) = |\{\mathbf{x} \in \Omega : u(\mathbf{x}) > q\}|$ is called the *distribution function* corresponding to u. Function m_u is, by definition, non-increasing and therefore admits a unique generalized inverse, called its *decreasing rearrangement*. This inverse takes the usual pointwise meaning when the function u has not flat regions, i.e. when $|\{\mathbf{x} \in \Omega : u(\mathbf{x}) = q\}| = 0$ for any $q \in \mathbb{R}$. In general, the decreasing rearrangement $u_* : \bar{\Omega}_* \to \mathbb{R}$ is given by:

$$u_*(s) = \begin{cases} \operatorname{ess\,sup}\{u(\mathbf{x}) : \mathbf{x} \in \Omega\} & \text{if } s = 0, \\ \inf\{q \in \mathbb{R} : m_u(q) \leq s\} & \text{if } s \in \Omega_*, \\ \operatorname{ess\,inf}\{u(\mathbf{x}) : \mathbf{x} \in \Omega\} & \text{if } s = |\Omega|. \end{cases}$$

Notice that since u_* is non-increasing in $\bar{\Omega}_*$, it is continuous but at most a countable subset of $\bar{\Omega}_*$. In particular, it is right-continuous for all $\sigma \in (0, |\Omega|]$.

The notion of rearrangement of a function is classical and was introduced by Hardy, Littlewood and Polya [15]. Applications include the study of isoperimetric and variational inequalities [20, 7, 17, 18], comparison of solutions of partial differential equations [28, 3, 30, 10, 11, 4], and others. We refer the reader to the monograph [21] for a extensive research on this topic.

Two of the most remarkable properties of the decreasing rearrangement are the equi-measurability property

$$\int_\Omega f(u(\mathbf{y}))d\mathbf{y} = \int_0^{|\Omega|} f(u_*(s))ds, \tag{6}$$

for any Borel function $f : \mathbb{R} \to \mathbb{R}_+$, and the contractivity

$$\|u_* - v_*\|_{L^p(\Omega_*)} \leq \|u - v\|_{L^p(\Omega)}, \tag{7}$$

for $u, v \in L^p(\Omega)$, $p \in [1, \infty]$.

For the extension of the decreasing rearrangement to families of functions depending on a parameter, e.g. $t \in [0, T]$, we first consider, for t fixed, the function $u(t) : \Omega \to \mathbb{R}$ given by $u(t)(\mathbf{x}) = u(t, \mathbf{x})$, for any $\mathbf{x} \in \Omega$. Then we define $u_* : (0, T) \times \Omega_* \to \mathbb{R}$ by $u_*(t, s) = u(t)_*(s)$.

3   Main results

Our first result ensures the well-posedness of problem $\mathrm{P}(\Omega, u_0)$ for $L^\infty(\Omega)$ initial data with bounded total variation. In addition, we show that the level sets structure of the solution is time invariant. Before stating our results, we collect here the main assumptions on the data problem, to which we shall refer to as **(H)**:

- $\Omega \subset \mathbb{R}^d$ is an open, bounded, and connected set $(d \geq 1)$.

- The final time, T, which simulate the time horizon of the diffusion process is a real, positive fixed number.

- The parameter λ is a real, nonnegative fixed number.

- $\mathcal{K} \in W^{1,\infty}(\mathbb{R})$ is nonnegative.

- $u_0 \in \mathcal{X} := L^\infty(\Omega) \cap BV(\Omega)$ is assumed to be, without loss of generality, non-negative.

Basic facts but also advanced results about the space of bounded variation $BV(\Omega)$ can be found in the book by Ambrosio et al. [5]. Notice that, depending on the space dimension $d \geq 2$ we have the continuous injections $BV(\Omega) \hookrightarrow L^{d/d-1}(\Omega)$. When $d = 1$ we have $\mathcal{X} \equiv BV(\Omega)$.

Theorem 1 *Assume (H). Then there exists a unique solution $u \in C^\infty([0,T]; \mathcal{X})$ of problem $P(\Omega, u_0)$. In addition, if $u_{01}, u_{02} \in \mathcal{X}$ and $u_1, u_2 \in C^\infty([0,T]; \mathcal{X})$ are the corresponding solutions to problems $P(\Omega, u_{01})$, $P(\Omega, u_{02})$ then*

$$\|u_1 - u_2\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|u_{01} - u_{02}\|_{L^2(\Omega)}, \tag{8}$$

for some constant $C > 0$.

Finally, suppose that $u_0(\mathbf{x}_1) = u_0(\mathbf{x}_2)$ for some $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$. Then $u(t, \mathbf{x}_1) = u(t, \mathbf{x}_2)$ for all $t \in (0, T]$.

Remark 1 *The existence and stability results of Theorem 1 may be extended to more general zero-order terms in the equation (1) of problem $P(\Omega, u_0)$. For instance, we can consider a function $f : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying $f(\cdot, \mathbf{x}, s) \in L^\infty(0,T)$, $f(t, \cdot, s) \in BV(\Omega)$, and $f(t, \mathbf{x}, \cdot) \in W^{1,\infty}(\mathbb{R})$. This regularity coincides with the initially obtained for the integral term of equation (1) in the approximation procedure to construct the solution. In addition, if $u_0(\mathbf{x}_1) = u_0(\mathbf{x}_2)$ implies $f(\cdot, \mathbf{x}_1, \cdot) = f(\cdot, \mathbf{x}_2, \cdot)$, then the time invariance of level sets holds.*

Replacing the set Ω by Ω_* and the initial data $u_0 \in \mathcal{X}$ by $v_0 \in \mathcal{X}_* \equiv BV(\Omega_*)$, Theorem 1 ensures the existence of a solution of problem $P(\Omega_*, u_{0*})$. Observe that $\Omega_* \subset \mathbb{R}$ is bounded because $\Omega \subset \mathbb{R}^d$ is bounded (assumption (H)) and this implies $BV(\Omega_*) \subset L^\infty(\Omega_*)$ and $\mathcal{X}_* \equiv BV(\Omega_*)$.

In the following result we obtain some properties of solutions of the one-dimensional problem $P(\Omega_*, v_0)$. Although the corollary is valid for any interval in $\mathbb{R}$, we keep the notation Ω_* for simplicity. The corresponding result for the discrete-time version, with $\lambda = 0$, of problem $P(\Omega_*, u_{0*})$ may be found in [12].

Corollary 1 *Assume (H), and let $v \in C^\infty([0,T]; \mathcal{X}_*)$ be the solution of problem $P(\Omega_*, v_0)$, for some nonincreasing $v_0 \in \mathcal{X}_*$. Then*

1. $\text{sign}(\partial_s v(t, \cdot)) = \text{sign}(\partial_s v_0) \leq 0$ a.e. in Ω_*, for all $t \in (0, T)$.

2. For $t > 0$, $v(t, 0) \leq v_0(0)$ and $v(t, |\Omega|) \geq v_0(|\Omega|)$.

3. If $\mathcal{K}$ is odd then $\int_{\Omega_*} v(t, s)ds = \int_{\Omega_*} v_0(s)ds$, for $t > 0$.

4. If $\mathcal{K} \in W^{m,\infty}(\mathbb{R})$ and $v_0 \in W^{m,p}(\Omega_*)$, for $m \in \mathbb{N}$ and $1 < p < \infty$, then $v \in C^\infty([0,T]; W^{m,p}(\Omega_*))$.

5. If $\lambda = 0$ and $\mathcal{K} \in C^1(\mathbb{R})$ is such that $\mathcal{K}'_h(\xi)\xi + \mathcal{K}_h(\xi) > 0$ then $v(t, \cdot) \to \text{const.}$ as $t \to \infty$.

Remark 2 *Condition in point 3 is a natural symmetry condition for convolution kernels and it is satisfied, for instance, by the Gaussian kernel. Condition in point 5 is also satisfied by the Gaussian kernel, if h is large enough.*

The next result establishes the connection between problems $P(\Omega, u_0)$ and $P(\Omega_*, u_{0*})$.

Theorem 2 *Assume (H). Then, $u \in C^\infty([0,T]; \mathcal{X})$ is a solution of $P(\Omega, u_0)$ if and only if $u_* \in C^\infty([0,T]; \mathcal{X}_*)$ is a solution of $P(\Omega_*, u_{0*})$.*

Theorem 2 implies that the solution of the multi-dimensional problem $P(\Omega, u_0)$ may be constructed by solving the one-dimensional problem $P(\Omega_*, u_{0*})$. Indeed, using the level sets invariance asserted in Theorem 1, we deduce

$$u(t, \mathbf{x}) = u_*(t, s) \quad \text{for a.e. } \mathbf{x} \in \{\mathbf{y} \in \Omega : u_0(\mathbf{y}) = u_{0*}(s)\},$$

for all $t \in [0, T]$. When image processing applications are considered, by property 1 of Corollary 1, the solution to $P(\Omega, u_0)$ may be understood as a *contrast change* of the initial image, u_0.

Indeed, this property also implies that if, initially, u_0 has no flat regions, and therefore u_{0*} is decreasing, then the solution of $P(\Omega_*, u_{0*})$ verifies this property for all $t > 0$. Then, Theorem 1 implies that the solution of $P(\Omega, u_0)$ has no flat regions for all $t > 0$.

The last theorem is an extension of a result given in [12] for the discrete-time formulation with $\lambda = 0$. In it, we deduce the asymptotic behavior of the solution u_* of problem $P(\Omega_*, u_{0*})$ (and thus of u of problem $P(\Omega, u_0)$) in terms of the window size parameter, h. Although we state it for the Gaussian kernel, more general choices are possible, see [12, Remark 2].

Theorem 3 *Assume (H) with $\mathcal{K}(\xi) = e^{-\xi^2}$ and $u_0 \in \mathcal{X}$ having no flat regions. Suppose, in addition, that $u_{0*} \in C^3(\bar{\Omega}_*)$. Then, for all $(t, s) \in [0, T] \times \Omega_*$, there exist positive constants α_1, α_2 independent of h such that the solution $u_* \in C^\infty([0, T]; C^3(\bar{\Omega}_*))$ of $P(\Omega_*, u_{0*})$ satisfies*

$$\partial_t u_*(t, s) = \lambda(u_{0*}(s) - u_*(t, s)) + \alpha_1 \tilde{k}_h(t, s) h^2 - \alpha_2 \frac{\partial_{ss}^2 u_*(t, s)}{|\partial_s u_*(t, s)|^3} h^3 + O(h^{7/2}), \tag{9}$$

with

$$\tilde{k}_h(t, s) = \frac{\mathcal{K}_h(u_*(t, |\Omega|) - u_*(t, s))}{|\partial_s u_*(t, |\Omega|)|} - \frac{\mathcal{K}_h(u_*(t, 0) - u_*(t, s))}{|\partial_s u_*(t, 0)|}, \tag{10}$$

and with $\alpha_1 \approx 1/(2\sqrt{\pi})$, and $\alpha_2 \approx 1$.

Two interesting effects captured by (9) are the following:

1. The border effect (range shrinking). Function $\tilde{k}_h$ is *active* only when s is close to the boundaries, $s \approx 0$ and $s \approx |\Omega|$. For $s \approx 0$, $\tilde{k}_h(t, s) < 0$ contributes to the decrease of the largest values of u_* while for $s \approx |\Omega|$ we have $\tilde{k}_h(t, s) > 0$, increasing the smallest values of u_*. Therefore, this term tends to flatten u_*. In image processing terms, a loss of contrast is induced.

2. The term

$$-\frac{\partial_{ss}^2 u_*(t, s)}{|\partial_s u_*(t, s)|^3}$$

is anti-diffusive, inducing large gradients on $u_*(t, \cdot)$ in a neighborhood of inflexion points. In this sense, the scheme (9) is related to the shock filter introduced by Álvarez and Mazorra [2]

$$v_t + F(G_\sigma v_{xx}, G_\sigma v_x) v_x = 0, \tag{11}$$

where G_σ is a smoothing kernel and function F satisfies $F(p,q)pq \geq 0$ for any $p, q \in \mathbb{R}$. Indeed, neglecting the fidelity, the border and the lower order terms, and defining $F(p,q) = \frac{p}{q|q|^3}$, we render (9) to the form (11).

This property can be exploited to produce a partition of the image so the model can be interpreted as a tool for fast segmentation and classification. An example is proposed in the numerical experiments where a time-continuous version of the NF is implemented.

4   Discretization and numerical examples

For the discretization of problem $P(\Omega, u_0)$, for $u_0 : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, we take advantage of the equivalence result stated in Theorem 2. Thus, we first calculate a numerical approximation, $\tilde{u}_{0*}$, to the decreasing rearrangement $u_{0*} : \Omega_* \subset \mathbb{R} \to \mathbb{R}$ and consider the problem $P(\Omega_*, \tilde{u}_{0*})$. Then, we discretize this one-dimensional problem and compute a numerical approximation, $v : [0,T] \times \Omega_* \to \mathbb{R}$. By Theorem 2, v is, in fact, an approximation to u_*, where $u : [0,T] \times \Omega \to \mathbb{R}$ is a solution to problem $P(\Omega, u_0)$. Then, we finally recover an approximation, $\tilde{u}$, to u by defining

$$\tilde{u}(t, \mathbf{x}) = v(t, s) \quad \text{for a.e. } \mathbf{x} \in \{\mathbf{y} \in \Omega : \tilde{u}_0(\mathbf{y}) = \tilde{u}_{0*}(s)\}. \tag{12}$$

Inspired by the image processing application of problem $P(\Omega, u_0)$, we consider a piecewise constant approximation to its solutions. Let $\Omega \subset \mathbb{R}^2$ be, for simplicity, a rectangle domain and consider a uniform mesh on Ω enclosing square elements (pixels), T_{mn}, of unit area, with barycenters denoted by $\mathbf{x}_{mn}$, for $m = 1, \ldots, M$ and $n = 1, \ldots, N$. Given $u_0 \in L^\infty(\Omega) \cap BV(\Omega)$, we consider its piecewise constant interpolator $\tilde{u}_0(\mathbf{x}) = u_0(\mathbf{x}_{mn})$ if $\mathbf{x} \in T_{mn}$.

The interpolator $\tilde{u}_0$ has a finite number, $Q \in \mathbb{N}$, of quantized levels that we denote by q_i, with $\max(\tilde{u}_0) = q_1 > \ldots > q_Q = \min(\tilde{u}_0)$. That is $\tilde{u}_0(\mathbf{x}) = \sum_{j=1}^Q q_j \chi_{E_j}(\mathbf{x})$, where E_j are the level sets of $\tilde{u}_0$,

$$E_j = \{\mathbf{x} \in \Omega : \tilde{u}_0(\mathbf{x}) = q_j\}, \quad j = 1, \ldots, Q.$$

Since $\tilde{u}_0$ is piecewise constant, the decreasing rearrangement of $\tilde{u}_0$ is piecewise constant too, and given by

$$\tilde{u}_{0*}(s) = \sum_{j=1}^Q q_j \chi_{I_j}(s), \tag{13}$$

with $I_j = [a_{j-1}, a_j)$ for $j = 1, \ldots, Q$, and $a_0 = 0$, $a_1 = |E_1|$, $a_2 = |E_1| + |E_2|$,...,$a_Q = \sum_{j=1}^Q |E_j| = |\Omega|$.

Let v be a candidate to solve problem $P(\Omega_*, \tilde{u}_{0*})$. Due to the time-invariance of the level sets structure of the solution to this problem, see Theorem 1, we may express v as

$$v(t, s) = \sum_{j=1}^Q c_j(t) \chi_{I_j}(s), \tag{14}$$

with $c_1(t) \geq \ldots \geq c_Q(t)$, for $t \in (0,T]$, $c_j(0) \equiv c_j^0 = q_j$, for $j = 1, \ldots, Q$. Substituting v in equation (1), we get, for $s \in I_j$ and $j = 1, \ldots, Q$,

$$c_j'(t) = \sum_{k=1}^Q \mathcal{K}_h(c_k(t) - c_j(t))(c_k(t) - c_j(t))\mu_k + \lambda(c_j^0 - c_j(t)), \tag{15}$$

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with $\mu_k = a_k - a_{k-1} = |E_k|$. Since, by assumptions (H), the right hand side of (15) is Lipschitz continuous, the existence and uniqueness of a smooth $\mathbf{c} = (c_1, \dots, c_Q) : [0, T] \to \mathbb{R}_+^Q$ satisfying (15) and $\mathbf{c}(0) = \mathbf{c}^0$ follows.

For the time discretization, we take a uniform mesh of the interval $[0, T]$ of size $\tau > 0$, and use the notation $\mathbf{c}^n = \mathbf{c}(t_n)$, with $t_n = n\tau$, and $n = 0, 1, 2, \dots$ Then, we consider the following implicit time discretization of problem (15). For $j = 1, \dots, Q$ and $n \geq 1$, solve

$$c_j^n = c_j^{n-1} + \tau \sum_{k=1}^Q \mathcal{K}_h(c_k^n - c_j^n)(c_k^n - c_j^n)\mu_k + \tau\lambda(c_j^0 - c_j^n). \tag{16}$$

Since problem (16) is a nonlinear algebraic system of equations, we use a fixed point argument to approximate its solution, $\mathbf{c}^n$, at each discrete time t_n, from the previous approximation $\mathbf{c}^{n-1}$. Let $\mathbf{c}^{n,0} = \mathbf{c}^{n-1}$. Then, for $m \geq 1$ the problem is to find $\mathbf{c}^{n,m}$ solving the linear system

$$c_j^{n,m} = c_j^{n-1} + \tau \sum_{k=1}^Q \mathcal{K}_h(c_k^{n,m-1} - c_j^{n,m-1})(c_k^{n,m} - c_j^{n,m})\mu_k + \tau\lambda(c_j^0 - c_j^{n,m}), \tag{17}$$

for $j = 1, \dots, Q$. We choose the stopping criterion $\|\mathbf{c}^{n,m} - \mathbf{c}^{n,m-1}\|_\infty < \text{tol}$, for values of tol chosen empirically, and then set $\mathbf{c}^n = \mathbf{c}^{n,m}$.

Finally, using formula (12), the expression of the initial datum (13), and the definition (14), we recover a piecewise constant approximation to the original problem, $\mathrm{P}(\Omega, u_0)$, taking

$$\tilde{u}(t, \mathbf{x}) = c_j^n \quad \text{if } t \in [t_n, t_{n+1}), \quad \mathbf{x} \in \{\mathbf{y} \in \Omega : \tilde{u}_0(\mathbf{y}) = q_j\}.$$

4.1   Example. Histogram based image segmentation

As an application we consider a Grand Challenge in Biomedical Image Analysis. This is a computer vision problem in biomedicine which consists of overlapping cells segmentation and subcellular nucleus and cytoplasm detection, see [26], [25]. The dataset was downloaded from the *Overlapping Cervical Cytology Image Segmentation Challenge*[1], ISBI 2014.

The data set is composed by 512×512 real and synthetic images containing two or more cells with different degrees of overlapping, contrast, and texture. The phantom images allow the quantitative analysis of segmentation procedures through their ground-truth, which is carried out by using the Dice similarity coefficient, DC: for two sets (images) A and B,

$$DC = \frac{2|A \cap B|}{|A| + |B|}.$$

Observe that values of DC close to one indicate high coincidence of the images, that is, of the ground-truth segmentation and the segmentation obtained with our method.

For running our algorithm, that is, providing an approximation, $\mathbf{c}^n$, of (15), we consider the usual number of image quantization levels, $Q = 256$. The fidelity term is ignored ($\lambda = 0$), and the range window parameter, h, is set as $h = 25$ for nucleus detection, and as $h = 5$ for cytoplasm detection. The tolerance in the fixed point loop (17) is taken as $\text{tol} = 1.e - 5$. As a

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[1] http://cs.adelaide.edu.au/~carneiro/isbi14_challenge/index.html

Table 1: Example: Segmentation results for some samples of test90 dataset.

Sample	1	15	30	45	60	75	90	All (mean)
Cytoplasm DC	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.98
Nucleus DC	0.93	0.94	0.82	0.88	0.90	0.81	0.85	0.87
Execution time	2.44	4.33	5.11	5.12	5.57	5.94	4.18	5.33

stopping criterion, we consider a combination of a maximum number of time iterations (1000), and an energy stabilization criterion,

$$|J(\mathbf{c}^n) - J(\mathbf{c}^{n-1})| < 1.e - 10,$$

where $J(\mathbf{c}^n)$ is the discrete version of the functional given by (5), for $w \equiv 1$ and $g(s) = \exp(-s/h)$. Finally, we implement a variable time step, $\tau(n)$, inspired by the proof of existence of solutions and given by, for $n \geq 2$,

$$\tau(n) = \frac{\tau(0)}{|J(\mathbf{c}^{n-1}) - J(\mathbf{c}^{n-2})|}$$

with $\tau(1) = \tau(0) = \left(\max_j \{ \sum_{k=1}^{Q} \mathcal{K}_h(c_k^0 - c_j^0)\mu_k \} \right)^{-1}$. In the experiments, we observed that $\tau(n)$ ranges from order 10^{-7} in the first iterations to order 10^{-1} just before convergence.

We summarize our results for the test90 dataset in Table 1, where we show the DC for some specific samples, and the mean DC of the ninety samples contained in the dataset. We may check that DC values are very high for the segmentation of both regions of interest (cytoplasm and nucleus), always above the range obtained in [26, 25]. The execution times are given for a Matlab implementation of the algorithm, running on a standard laptop (Intel Core i7-2.80 GHz processor, 8GB RAM).

In Figure 1, we show the segmentation process for the two regions of interest. The first column corresponds to the initial image. The second column, to the background extraction, and the third column to the nucleus segmentation. Thus, the cytoplasm is the difference between the images shown in the third and second column. Finally, the fourth column shows the difference between the ground-truth nucleus segmentation and the obtained with our method.

5   Proofs

Proof of Theorem 1. We divide the proof in several steps.
Step 1. Existence of a local in time solution to an auxiliary problem with smooth data.

We assume $u_0 \in W^{1,\infty}(\Omega)$, and consider the following auxiliary problem, obtained using the change of unknown $u = we^{\mu t}$ in (1), for some positive constant μ to be fixed:

$$\partial_t w(t,\mathbf{x}) = \int_\Omega \mathcal{K}_h\big(e^{\mu t}(w(t,\mathbf{y}) - w(t,\mathbf{x}))\big)(w(t,\mathbf{y}) - w(t,\mathbf{x}))d\mathbf{y}$$
$$+ \lambda(w_0(\mathbf{x}) - w(t,\mathbf{x})) - \mu w(t,\mathbf{x}), \tag{18}$$

for $(t,\mathbf{x}) \in (0,T_0) \times \Omega$, and for the initial data $w(0,\cdot) = w_0 = u_0 \in W^{1,\infty}(\Omega)$. Here, $T_0 > 0$ will be fixed later.

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Time discretization. Let $N \in \mathbb{N}$, $\tau = T_0/N$ and $t_j = j\tau$, for $j = 0, \ldots, N$. Assume that $w_j \in W^{1,\infty}(\Omega)$ is given and consider the functional $A : L^\infty(\Omega) \to L^\infty(\Omega)$ given by

$$
A(\varphi(\mathbf{x})) = \frac{1}{1 + \tau(\lambda + \mu)} \Big(w_j(\mathbf{x}) + \tau \int_{\Omega_*} \mathcal{K}_h\big(e^{\mu t_j}(w_j(\mathbf{y}) - w_j(\mathbf{x}))\big)(\varphi(\mathbf{y}) - \varphi(\mathbf{x}))d\mathbf{y}
$$
$$
+ \tau\lambda w_0(\mathbf{x}) \Big),
$$

for $\mathbf{x} \in \Omega$. Observe that if A has a fixed point φ, then we may define $w_{j+1} = \varphi$ to get the following semi-implicit version of (18)

$$
w_{j+1}(\mathbf{x}) = w_j(\mathbf{x}) + \tau \int_\Omega \mathcal{K}_h\big(e^{\mu t_j}(w_j(\mathbf{y}) - w_j(\mathbf{x}))\big)(w_{j+1}(\mathbf{y}) - w_{j+1}(\mathbf{x}))d\mathbf{y}
$$
$$
+ \tau\lambda(w_0(\mathbf{x}) - w_{j+1}(\mathbf{x})) - \tau\mu w_{j+1}(\mathbf{x}). \tag{19}
$$

We have,

$$
|A(\varphi(\mathbf{x})) - A(\psi(\mathbf{x}))| = \frac{\tau}{1 + \tau\mu} \int_\Omega \mathcal{K}_h\big(e^{\mu t_j}(w_j(\mathbf{y}) - w_j(\mathbf{x}))\big)
$$
$$
\times \big|\varphi(\mathbf{y}) - \psi(\mathbf{y}) - (\varphi(\mathbf{x}) - \psi(\mathbf{x}))\big|d\mathbf{y}
$$
$$
\leq \frac{2\tau|\Omega|}{1 + \tau\mu}\|\mathcal{K}_h\|_\infty\|\varphi - \psi\|_\infty.
$$

Therefore, for $\mu > 2|\Omega|\|\mathcal{K}_h\|_\infty$, the mapping A is contractive in $L^\infty(\Omega_*)$, and a unique fixed point, w_{j+1} verifying (19) does exist.

We have the following uniform estimates for w_{j+1}. One one hand, from (19) we obtain

$$
\|w_{j+1}\|_\infty \leq \frac{1}{1 + \tau(\lambda + \mu - 2|\Omega|\|\mathcal{K}_h\|_\infty)}\Big(\|w_j\|_\infty + \tau\lambda\|w_0\|_\infty\Big),
$$

which gives (recall $\mu > 2|\Omega|\|\mathcal{K}_h\|_\infty$) the uniform estimate

$$
\|w_{j+1}\|_\infty \leq M_0 \tag{20}
$$

with M_0 depending only on $\|w_0\|_\infty$.

On the other hand, since $w_0, w_j \in W^{1,\infty}(\Omega)$, we deduce from (19) $w_{j+1} \in W^{1,\infty}(\Omega)$. This regularity allows to differentiate in (19) with respect to the $k-$th component of $\mathbf{x}$, denoted by x_k, to obtain for a.e. $\mathbf{x} \in \Omega$,

$$
F_1(\mathbf{x})\frac{\partial w_{j+1}}{\partial x_k}(\mathbf{x}) = F_2(\mathbf{x})\frac{\partial w_j}{\partial x_k}(\mathbf{x}) + \tau\lambda\frac{\partial w_0}{\partial x_k}(\mathbf{x}),
$$

with

$$
F_1(\mathbf{x}) = 1 + \tau\Big(\lambda + \mu + \int_\Omega \mathcal{K}_h\big(e^{\mu t_j}(w_j(\mathbf{y}) - w_j(\mathbf{x}))\big)d\mathbf{y}\Big),
$$
$$
F_2(\mathbf{x}) = 1 - \tau e^{\mu t_j} \int_\Omega \mathcal{K}_h'\big(e^{\mu t_j}(w_j(\mathbf{y}) - w_j(\mathbf{x}))\big)(w_{j+1}(\mathbf{y}) - w_{j+1}(\mathbf{x}))d\mathbf{y},
$$

from where we deduce

$$\|\nabla w_{j+1}\|_\infty \leq \frac{1}{1+\tau(\lambda+\mu)}\Big(\tau\lambda\|\nabla w_0\|_\infty + \big(1+2\tau e^{\mu t_j}|\Omega|M_0\|\mathcal{K}_h'\|_\infty\big)\|\nabla w_j\|_\infty\Big).$$

Solving this differences inequality, we find that, by redefining μ to satisfy $\mu > 2e^{\mu T_0}|\Omega|\|\mathcal{K}_h'\|_\infty M_0$, we obtain the uniform estimate $\|\nabla w_{j+1}\|_\infty \leq M_1$, with M_1 depending only on $\|\nabla w_0\|_\infty$. This election of μ is possible by restricting T_0 to be

$$T_0 < \frac{1}{\mu}\log\frac{\mu}{2|\Omega|\|\mathcal{K}_h'\|_\infty M_0}. \tag{21}$$

Time interpolators and passing to the limit $\tau \to 0$. We define, for $(t,\mathbf{x}) \in (t_j, t_{j+1}] \times \Omega$, the piecewise constant and piecewise linear interpolators

$$w^{(\tau)}(t,\mathbf{x}) = w_{j+1}(\mathbf{x}), \quad \tilde{w}^{(\tau)}(t,\mathbf{x}) = w_{j+1}(\mathbf{x}) + \frac{t_{j+1}-t}{\tau}(w_j(\mathbf{x}) - w_{j+1}(\mathbf{x})).$$

Using the uniform L^∞ estimates of w_{j+1} and ∇w_{j+1}, we deduce the corresponding uniform estimates for $\|\nabla w^{(\tau)}\|_{L^\infty(Q_{T_0})}$, $\|\nabla \tilde{w}^{(\tau)}\|_{L^\infty(Q_{T_0})}$ and $\|\partial_t \tilde{w}^{(\tau)}\|_{L^\infty(Q_{T_0})}$, implying the existence of $w \in L^\infty(0,T_0;W^{1,\infty}(\Omega))$ and $\tilde{w} \in W^{1,\infty}(Q_{T_0})$ such that, at least in a subsequence (not relabeled), as $\tau \to 0$,

$$w^{(\tau)} \to w \quad \text{weakly* in } L^\infty(0,T_0;W^{1,\infty}(\Omega)),$$
$$\tilde{w}^{(\tau)} \to \tilde{w} \quad \text{weakly* in } W^{1,\infty}(Q_{T_0}). \tag{22}$$

In particular, by compactness

$$\tilde{w}^{(\tau)} \to \tilde{w} \quad \text{uniformly in } C([0,T_0] \times \bar{\Omega}).$$

Since, for $t \in (t_j, t_{j+1}]$,

$$|w^{(\tau)}(t,\mathbf{x}) - \tilde{w}^{(\tau)}(t,\mathbf{x})| = |\frac{(j+1)\tau - t}{\tau}(w_j(\mathbf{x}) - w_{j+1}(\mathbf{x}))| \leq \tau\|\partial_t \tilde{w}^{(\tau)}\|_{L^\infty(Q_{T_0})},$$

we deduce both $w = \tilde{w}$ and

$$w^{(\tau)} \to w \quad \text{uniformly in } C([0,T_0] \times \bar{\Omega}). \tag{23}$$

Considering the shift operator $\sigma_\tau w^{(\tau)}(t,\cdot) = w_j$, and introducing the approximation $e_\tau^{\mu t} = e^{\mu t_j}$, for $t \in (t_j, t_{j+1}]$, we may rewrite (19) as

$$\partial_t \tilde{w}^{(\tau)}(t,\mathbf{x}) = \int_\Omega \mathcal{K}_h\big(e_\tau^{\mu t}(w^{(\tau)}(t,\mathbf{y}) - w^{(\tau)}(t,\mathbf{x}))\big)(w^{(\tau)}(t,\mathbf{y}) - w^{(\tau)}(t,\mathbf{x}))d\mathbf{y}$$
$$+ \lambda(w_0(\mathbf{x}) - w^{(\tau)}(t,\mathbf{x})) - \mu w^{(\tau)}(t,\mathbf{x}), \tag{24}$$

and due to the convergence properties (22) and (23), we may pass to the limit $\tau \to 0$ in (24) to deduce that w is a solution of (18).

Continuation of the solution to an arbitrary time T. Given the solution, w, of problem (18) in Q_{T_0}, we may consider the same problem for the initial datum $w(T_0, \cdot)$. Since $w(T_0, \cdot) \in W^{1,\infty}(\Omega)$ and the constant $T_0 > 0$ only depends on $|\Omega|$, $\|\mathcal{K}_h'\|_\infty$ and $\|u_0\|_\infty$, see (20) and (21), we obtain a

new solution $w \in C(T_0, 2T_0; W^{1,\infty}(\Omega))$. Clearly, this procedure may be extended to an arbitrarily fixed T. Once this is done, a boot-strap argument allows us to deduce $w \in C^\infty(0, T; W^{1,\infty}(\Omega))$, implying that $u = we^{\mu t} \in C^\infty([0,T]; W^{1,\infty}(\Omega))$ is a solution of $P(\Omega, u_0)$ in Q_T.

Step 2. Non smooth initial data.

Let us consider a sequence $u_{0\varepsilon} \in C^\infty(\bar{\Omega})$ such that, as $\varepsilon \to 0$,

$$u_{0\varepsilon} \to u_0 \quad \text{in } L^\infty(\Omega) \tag{25}$$

$$\|\nabla u_{0\varepsilon}\|_{L^1(\Omega)} \to \mathrm{TV}(u_0), \tag{26}$$

where TV denotes total variation with respect to the $\mathbf{x}$ variable. Let us denote by u_ε to the corresponding solution of $P(\Omega, u_{0\varepsilon})$.

First, notice that u_ε is uniformly bounded in $L^\infty(Q_T)$ with respect to ε as a consequence of estimate (20) and property (25). We then obtain directly from equation (1) that

$$\partial_t u_\varepsilon \quad \text{is uniformly bounded in } L^\infty(Q_T). \tag{27}$$

Since $u_{0\varepsilon}$ is smooth, we may deduce an L^∞ bound for ∇u_ε as in Step 1, not necessarily uniform in ε, but which allows us to differentiate equation (1) with respect to x_k. After integration in $(0, t)$, we obtain

$$\frac{\partial u_\varepsilon}{\partial x_k}(t, \mathbf{x}) = \frac{\partial u_{0\varepsilon}}{\partial x_k}(\mathbf{x}) G_\varepsilon(t, \mathbf{x}) \Big(1 + \lambda \int_0^t (G_\varepsilon(\tau, \mathbf{x}))^{-1} d\tau \Big), \tag{28}$$

with $G_\varepsilon(t, \mathbf{x}) = \exp\big(-\int_0^t \big(\lambda + \eta_\varepsilon(\tau, \mathbf{x})\big) d\tau \big)$, and

$$\eta_\varepsilon(t, \mathbf{x}) = \int_\Omega \Big(\mathcal{K}_h'\big(u_\varepsilon(t, \mathbf{y}) - u_\varepsilon(t, \mathbf{x})\big)\big(u_\varepsilon(t, \mathbf{y}) - u_\varepsilon(t, \mathbf{x})\big) \tag{29}$$

$$+ \mathcal{K}_h\big(u_\varepsilon(t, \mathbf{y}) - u_\varepsilon(t, \mathbf{x})\big)\Big) d\mathbf{y}.$$

Since $\mathcal{K}_h \in W^{1,\infty}(\mathbb{R})$, we have η_ε uniformly bounded in $L^\infty(Q_T)$ and so G_ε and G_ε^{-1}. Therefore, using (26) we deduce from (28) that

$$\nabla u_\varepsilon \quad \text{is uniformly bounded in } L^\infty(0, T; L^1(\Omega)). \tag{30}$$

Bounds (27) and (30) allow to deduce, using the compactness result [24, Cor. 4, p. 85], the existence of $u \in C([0,T]; \mathcal{X})$ such that $u_\varepsilon \to u$ strongly in $L^p(Q_T)$, for all $p < \infty$, and a.e. in Q_T.

Similarly to the smooth case, this convergence allows to pass to the limit $\varepsilon \to 0$ in (1) (with u replaced by u_ε) and identify the limit u as a solution of $P(\Omega, u_0)$. Again, the property $u \in L^\infty(Q_T)$ and a boot-strap argument leads to $u \in C^\infty(0, T; \mathcal{X})$.

Stability and uniqueness. Let u_{01}, $u_{02} \in BV(\Omega)$ and u_1, $u_2 \in C^\infty([0,T]; \mathcal{X})$ be the corresponding solutions to problems $P(\Omega, u_{01})$, $P(\Omega, u_{02})$. Set $u = u_1 - u_2$ and $u_0 = u_{10} - u_{20}$. Then u satisfies

$$\partial_t u(t, \mathbf{x}) = \int_\Omega \Big(\Phi(u_1(t, \mathbf{y}) - u_1(t, \mathbf{x})) - \Phi(u_2(t, \mathbf{y}) - u_2(t, \mathbf{x})) \Big) d\mathbf{y}$$

$$+ \lambda(u_0(\mathbf{x}) - u(t, \mathbf{x})),$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}),$$

for $(t, \mathbf{x}) \in Q_T$, with $\Phi(s) = \mathcal{K}_h(s)s$. Multiplying this equation by u, integrating in Ω and using the Lipschitz continuity of Φ (with constant C_L) and Young's inequality, we deduce

$$\partial_t \int_\Omega |u(t, \mathbf{x})|^2 d\mathbf{x} \leq C_L \int_\Omega \int_\Omega |u(t, \mathbf{y}) - u(t, \mathbf{x})| u(t, \mathbf{x}) d\mathbf{y} d\mathbf{x} + \frac{\lambda}{2} \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x}$$

$$- \frac{\lambda}{2} \int_\Omega |u(t, \mathbf{x})|^2 d\mathbf{x}$$

$$\leq C_L \left(\int_\Omega |u(t, \mathbf{x})| d\mathbf{x} \right)^2 + \left(C_L - \frac{\lambda}{2} \right) \int_\Omega |u(t, \mathbf{x})|^2 d\mathbf{x}$$

$$+ \frac{\lambda}{2} \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x}.$$

Finally, using Jensen's and Gronwall's inequalities, we deduce $\|u(t, \cdot)\|_{L^2(\Omega)} \leq C \|u_0\|_{L^2(\Omega)}$ for all $t \in (0, T)$, and the result follows.

Time invariance of the level sets. The proof of this property is similar to the proof of the stability property. Let $u_0 \in \mathcal{X}$ and $u \in C^\infty([0, T]; \mathcal{X})$ be the corresponding solution to problem $P(\Omega, u_0)$. Assume $u_0(\mathbf{x}_1) = u_0(\mathbf{x}_2)$, and set $u_i(t) = u(t, \mathbf{x}_i)$, $i = 1, 2$. Then, from equation (1) we get

$$\partial_t(u_1(t) - u_2(t)) = \int_\Omega \Big(\Phi(u(t, \mathbf{y}) - u_1(t)) - \Phi(u(t, \mathbf{y}) - u_2(t)) \Big) d\mathbf{y}$$

$$- \lambda(u_1(t)) - u_2(t)).$$

Then, the Lipschitz continuity of Φ and Gronwall's lemma allow us to deduce the result. $\square$

Proof of Corollary 1. To prove point 1, notice that from (28) (in dimension $d = 1$) we deduce

$$\text{sign}(\partial_s v_\varepsilon) = \text{sign}(v'_{0\varepsilon}) \leq 0 \quad \text{a.e. in } (0, T) \times \Omega_*,$$

a property that also holds in the limit $\varepsilon \to 0$. Point 2 of the theorem follows from evaluating equation (1) in $s = 0$ and $s = |\Omega|$, using that $v(t, \cdot)$ is decreasing for all $t > 0$, and Gronwall's inequality. Point 3 is a consequence of the assumption on the symmetry of $\mathcal{K}$, under which the integral term in (1) vanishes when it is integrated in Ω_*. Point 4 is easily deduced by successive derivation of (28) (which also holds for $\varepsilon = 0$, under regularity assumptions). Point 5 is again deduced from (28) and the decreasing character of v_ε and v. Since, $TV(v_\varepsilon(t, \cdot)) \to TV(v(t, \cdot))$ and, using point 2, $TV(v(t, \cdot)) \leq c$ for all $t \geq 0$, we have that the integral term in (29) is evaluated inside a closed interval. Therefore, using the assumptions of point 5, we get $\eta_\varepsilon(t, s) > c_2 > 0$ uniformly in (t, s). Finally, we obtain the result from (28) in the limit $t \to \infty$ and $\varepsilon \to 0$. $\square$

Proof of Theorem 2. We split the proof in two steps.

Step 1. First we treat the case in which u_0 has no flat regions, that is when $|\{\mathbf{x} \in \Omega : u(\mathbf{x}) = q\}| = 0$ for any $q \in \mathbb{R}$. By the invariance of the level sets structure proven in Theorem 1 we deduce that neither the solution u of $P(\Omega, u_0)$ has flat regions. Then $m_u(t, \cdot)$ and $u_*(t, \cdot)$ are strictly decreasing, implying $u_*(t, m_u(t, q)) = q$ for any $q \in \mathbb{R}$. According to [21, Theorem 9.2.1], we have $\partial_t u_* = \partial_s \varphi$ where

$$\varphi(t, s) = \int_{\{u(t) > u_*(t,s)\}} \frac{\partial u}{\partial t}(t, \mathbf{x}) d\mathbf{x}, \tag{31}$$

and we used the notation $\{u(t) > u_*(t,s)\} = \{\mathbf{y} \in \Omega : u(t,\mathbf{y}) > u_*(t,s)\}$. Integrating (1) in $\{u(t) > u_*(t,s)\}$ we get

$$\varphi(t,s) = \int_{\{u(t)>u_*(t,s)\}} \int_\Omega \mathcal{K}_h(u(t,\mathbf{y}) - u(t,\mathbf{x}))(u(t,\mathbf{y}) - u(t,\mathbf{x}))d\mathbf{y}d\mathbf{x} \tag{32}$$

$$+ \lambda \int_{\{u(t)>u_*(t,s)\}} u_0(\mathbf{x})d\mathbf{x} - \lambda \int_{\{u(t)>u_*(t,s)\}} u(t,\mathbf{x})d\mathbf{x} = I_1 + I_2 + I_3.$$

Due to the u and u_* level sets equi-measure, it is immediate that

$$I_3 = -\lambda \int_0^s u_*(t,\sigma)d\sigma. \tag{33}$$

The equi-measurability property (6) implies

$$I_1 = \int_{\{u(t)>u_*(t,s)\}} \int_{\Omega_*} \mathcal{K}_h(u_*(t,\sigma) - u(t,\mathbf{x}))(u_*(t,\sigma) - u(t,\mathbf{x}))d\sigma d\mathbf{x},$$

from where we deduce

$$I_1 = \int_0^s \int_{\Omega_*} \mathcal{K}_h(u_*(t,\sigma) - u_*(t,\tau))(u_*(t,\sigma) - u_*(t,\tau))d\sigma d\tau. \tag{34}$$

To deal with the term I_2 we observe that due to the invariance of the level set structure, as stated in Theorem 1, we have that, for all $t \in [0,T]$ and $s \in \bar{\Omega}_*$, there exists $\alpha \in \bar{\Omega}_*$ such that

$$\{\mathbf{x} \in \Omega : u(t,\mathbf{x}) > u_*(t,s)\} = \{\mathbf{x} \in \Omega : u_0(\mathbf{x}) > u_{0*}(\alpha)\}.$$

Recalling that u and u_0 have not flat regions and taking the measure of these sets we deduce $s = \alpha$. Therefore,

$$I_2 = \lambda \int_{\{u_0>u_{0*}(s)\}} u_0(\mathbf{x})d\mathbf{x} = \int_0^s u_{0*}(\sigma)d\sigma. \tag{35}$$

Finally, substituting in identity (32) the expressions (31), (33), (34) and (35), and differentiating with respect to s, we deduce the result.

Conversely, let v be a solution of $P(\Omega_*, u_{0*})$. Since u_0 has not flat regions, $u'_{0*} < 0$ in Ω_*, and by point 1 of Corollary 1 we have $\partial_s v(t,s) < 0$ in $[0,T] \times \Omega_*$. We define

$$u(t,\mathbf{x}) = v(t,s) \quad \text{for a.e. } \mathbf{x} \in L(s) = \{\mathbf{y} \in \Omega : u_0(\mathbf{y}) = u_{0*}(s)\}, \tag{36}$$

and for all $t \in [0,T]$. Observe that since u_0 has not flat regions, we have $|L(s)| = 0$ for all s. Therefore, since $\partial_s v < 0$, we also deduce that u has not flat regions. By construction,

$$|\{\mathbf{x} \in \Omega : u(t,\mathbf{x}) > v(t,s)\}| = |\{\mathbf{x} \in \Omega : u_0(\mathbf{x}) > u_{0*}(s)\}| = s,$$

implying $u_* = v$. Differentiating in (36) with respect to t and using that v is a solution of $P(\Omega_*, u_{0*})$, we get, for $\mathbf{x} \in L(s)$,

$$\partial_t u(t,\mathbf{x}) = \partial_t v(t,s) = \int_{\Omega_*} \mathcal{K}_h(v(t,\sigma) - v(t,s))(v(t,\sigma) - v(t,s))d\sigma + \lambda(v_0(s) - v(t,s))$$

$$= \int_{\Omega_*} \mathcal{K}_h(u_*(t,\sigma) - u(t,\mathbf{x}))(u_*(t,\sigma) - u(t,\mathbf{x}))d\sigma + \lambda(u_0(\mathbf{x}) - u(t,\mathbf{x}))$$

$$= \int_\Omega \mathcal{K}_h(u(t,\mathbf{y}) - u(t,\mathbf{x}))(u(t,\mathbf{y}) - u(t,\mathbf{x}))d\mathbf{y} + \lambda(u_0(\mathbf{x}) - u(t,\mathbf{x})),$$

where we have used again the equi-measurability property (6).

Step 2. We now treat the general case in which $u_0 \in \mathcal{X}$ may have flat regions. We use the following lemma.

Lemma 1 *Let $u_0 \in \mathcal{X}$. Then there exists a sequence $u_{0j} \in \mathcal{X}$ such that u_{0j} has no flat regions and $u_{0j} \to u_0$ in $\mathcal{X}$.*

We may then apply the Step 1 of this proof to each u_{0j} to obtain that $u_j \in C^\infty([0,T]; \mathcal{X})$ is a solution of $P(\Omega, u_{0j})$ (without flat regions) if and only if $(u_j)_* \in C^\infty(0,T; \mathcal{X}_*)$ is a solution of $P(\Omega_*, (u_{0j})_*)$. Now we perform the limit $j \to \infty$.

Let $u \in C^\infty([0,T]; \mathcal{X})$ and $v \in C^\infty(0,T; \mathcal{X}_*)$ be the solutions of problems $P(\Omega, u_0)$ and $P(\Omega_*, u_{0*})$ ensured by Theorem 1.

Using the strong continuity of the decreasing rearrangement operation in $L^2(\Omega)$, see (7), and the stability property (8) applied to problem $P(\Omega, u_0)$, we obtain

$$\|u_* - (u_j)_*\|_{L^\infty(0,T;L^2(\Omega_*))} \le \|u - u_j\|_{L^\infty(0,T;L^2(\Omega))} \le C_1 \|u_0 - u_{0j}\|_{L^2(\Omega)}.$$

The same arguments in reverse order applied to problem $P(\Omega_*, u_{0*})$ leads to

$$\|v - (u_j)_*\|_{L^\infty(0,T;L^2(\Omega_*))} \le C_2 \|u_{0*} - (u_{0j})_*\|_{L^2(\Omega_*)} \le C_2 \|u_0 - u_{0j}\|_{L^2(\Omega)}.$$

Therefore, using the triangle inequality we deduce

$$\|v - u_*\|_{L^\infty(0,T;L^2(\Omega_*))} \le (C_1 + C_2) \|u_0 - u_{0j}\|_{L^2(\Omega)} \to 0,$$

as $j \to \infty$. $\square$

Proof of Lemma 1.

In this proof, we rename u_0 by u and u_{0j} by u_j. Let, for $i \in I$, $E_i = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = q_i\}$ with $|E_i| > 0$, be the collection of flat regions of u which is, at most, countable. Thus, $I \subset \mathbb{N}$. Let χ_{E_i} and $P(E_i)$ denote the characteristic function of the set E_i and its perimeter, respectively. We consider the functions

$$\varphi_i^j(\mathbf{x}) = \frac{\chi_{E_i}(\mathbf{x}) \min(q_i - q_{i+1}, 1)}{i^2 \big(j(1 + P(E_i)) + v(\mathbf{x})\big)},$$

where $v \in BV(\Omega)$ is a non-negative function without flat regions. Observe that since $u \in BV(\Omega)$ we have $P(E_i) < \infty$ for all $i \in I$.

Consider, for $j \in \mathbb{N}$, the sequence of $L^\infty(\Omega)$ functions

$$u_j(\mathbf{x}) = u(\mathbf{x}) - \sum_{i \in I} \varphi_i^j(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \setminus \cup_{i \in I} E_i \\ q_i - \frac{\min(q_i - q_{i+1}, 1)}{i^2 \big(j(1 + P(E_i)) + v(\mathbf{x})\big)} & \text{if } \mathbf{x} \in E_i, \text{for some } i \in I. \end{cases}$$

We have:

(1) *u_j has no flat regions in Ω.* Let $q \in \mathbb{R}$. We use the decomposition

$$\{\mathbf{x} \in \Omega : u_j(\mathbf{x}) = q\} = \{\mathbf{x} \in \Omega \setminus \cup_i E_i : u(\mathbf{x}) = q\} \bigcup \cup_i \{\mathbf{x} \in E_i : u_j(\mathbf{x}) = q\}.$$

If $q = q_i$ for some $i \in I$ then $\mathbf{x} \in E_i$, and, by definition, $u_j(\mathbf{x}) = q_i$ if

$$\frac{1}{i^2 \big(j(1 + P(E_i)) + v(\mathbf{x})\big)} = 0,$$

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which is not possible. Therefore, if $q = q_i$ we have $|u_j = q_i| = 0$. If $q \neq q_i$ for all $i \in I$ then $|\{\mathbf{x} \in \Omega \setminus \cup_i E_i : u(\mathbf{x}) = q\}| = 0$, so

$$|u_j(\mathbf{x}) = q| = |v(\mathbf{x}) = -j(1 + P(E_i)) + \min(q_i - q_{i+1}, 1)/(i^2(q_i - q))| = 0,$$

since v has no flat regions.

(2) $u_j \to u$ in $L^p(\Omega)$ for any $1 \leq p \leq \infty$. This is immediate, since $|u(\mathbf{x}) - u_j(\mathbf{x})| \leq \frac{1}{j}$.

(3) $u_j \in BV(\Omega)$ and $u_j \to u$ in $BV(\Omega)$. According to [5, Proposition 3.38], for each $i \in I$, we can find a sequence $w_h^i \in C^\infty(\Omega)$ with $0 \leq w_h^i \leq 1$ such that $w_h^i \to \chi_{E_i}$ in $L^1(\Omega)$ as $h \to 0$, and

$$\lim_{h \to 0} \int_\Omega |\nabla w_h^i| = TV(\chi_{E_i}) = P(E_i) < \infty,$$

since $u \in BV(\Omega)$. We also introduce a regularizing sequence $v_h \in C^\infty(\Omega)$ such that $v_h > 0$ and $v_h \to v$ in $BV(\Omega)$. Let $g_h^i = w_h^i \min(q_i - q_{i+1}, 1)/(i^2(j(1 + P(E_i)) + v_h))$. Then,

$$\int_\Omega |\nabla g_h^i| \leq \frac{1}{i^2 j(1 + P(E_i))} \int_\Omega |\nabla w_h^i| + \frac{1}{(ij(1 + P(E_i)))^2} \int_\Omega |\nabla v_h|, \tag{37}$$

implying that g_h^i is uniformly bounded in $BV(\Omega)$ with respect to h. Therefore, there exists $g^i \in BV(\Omega)$ and a a subsequence of g_h^i (not relabeled) such that $g_h^i \to g^i$ strongly in $L^1(\Omega)$ as $h \to 0$. Since, by the Dominated Convergence Theorem we have $g_h^i \to \varphi_i^j$ in $L^1(\Omega)$ as $h \to 0$, we deduce $g^i = \varphi_i^j \in BV(\Omega)$. Taking the limit $h \to 0$ in (37) we get

$$TV(\varphi_i^j) \leq \frac{P(E_i)}{i^2 j(1 + P(E_i))} + \frac{1}{(ij(1 + P(E_i)))^2} \int_\Omega |\nabla v_h| \leq \frac{c}{i^2 j}, \tag{38}$$

with $c > 0$ independent of i and j. Thus, using the definition of u_j, the triangle inequality, and (38) we get

$$TV(u_j) \leq TV(u) + \sum_{i \in I} TV(\varphi_i^j) \leq TV(u) + \frac{c}{j}.$$

Therefore, $u_j \in BV(\Omega)$. Finally, from the definition of u_j and (38) we have

$$TV(u - u_j) \leq \sum_{i \in I} TV(\varphi_i^j) \to 0 \quad \text{as } j \to \infty.$$

$\square$

Proof of Theorem 3. Define

$$I(t, s) = \int_{\Omega_*} \mathcal{K}_h(u_*(t, \sigma) - u_*(t, s))(u_*(t, \sigma) - u_*(t, s)) d\sigma. \tag{39}$$

Since u_0 has not flat regions, we have $u_{0*}' < 0$. Then, due to points 1 and 2 of Corollary 1 we have $\partial_s u_* < 0$ in $[0, T) \times \Omega_*$, and $u_*(t, \Omega_*) \subset u_{0*}(\Omega_*)$ for all $t \in [0, T]$, respectively.

Let us consider the inverse of $u_*(t, \cdot)$, the distribution function of u, $m_u(t, \cdot)$. Using the change of variable $s = m_u(\cdot, z)$ and writing $\sigma = m_u(\cdot, q)$, we obtain from (39)

$$I_1(t, z) := I(t, m_u(t, z)) = \int_{u_*(t, |\Omega|)}^{u_*(t, 0)} \mathcal{K}_h(q - z)(q - z) \frac{dq}{|\partial_s u_*(t, m_u(t, q))|}. \tag{40}$$

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Using the explicit form of $\mathcal{K}$ and integrating by parts, we obtain

$$I_1(t,z) = \frac{h^2}{2}\left(\tilde{k}_h(m_u(t,z)) + \int_{u_*(t,|\Omega|)}^{u_*(t,0)} \mathcal{K}_h(q-z)\frac{\partial_{ss}^2 u_*(t,m_u(t,q))}{(\partial_s u_*(t,m_u(t,q)))^3}dq\right), \qquad (41)$$

with $\tilde{k}_h$ given by (10).

By assumption, function

$$f(t,q) = \frac{\partial_{ss}^2 u_*(t,m_u(t,q))}{(\partial_s u_*(t,m_u(t,q)))^3}$$

is bounded in $[u_*(t,(|\Omega|)), u_*(t,0)]$ and by point 4 of Corollary 1 it is continuously differentiable in $(u_*(t,|\Omega|), u_*(t,0))$.

Consider the interval $J_h = \{q : |q-z| < \sqrt{h}\}$. By well known properties of the Gaussian kernel, we have

$$\kappa(h) := \int_{J_h} \mathcal{K}_h(q-z)dq < \int_{\mathbb{R}} \mathcal{K}_h(q)dq = h\sqrt{\pi}, \qquad (42)$$

and

$$\mathcal{K}_h(z-q) \le e^{-1/h} \quad \text{if} \quad q \in J_h^C = \{q : |q-z| \ge \sqrt{h}\}. \qquad (43)$$

In particular, from (43) we get

$$\left|\int_{J_h^C} \mathcal{K}_h(q-z)f(t,q)dq\right| < O(h^\alpha) \quad \text{for any } \alpha > 0. \qquad (44)$$

Taylor's formula implies

$$\int_{u_*(|\Omega|)}^{u_*(0)} \mathcal{K}_h(q-z)f(t,q)dq = \int_{J_h} \mathcal{K}_h(q-z)(f(t,z)+O(\sqrt{h}))dq$$
$$+ \int_{J_h^C} \mathcal{K}_h(q-z)f(t,q)dq.$$

Therefore, from (41), (44) and (42) we deduce, using $\partial_s u_* < 0$,

$$I_1(t,z) = \frac{h^2}{2}\left(\tilde{k}_h(m_u(t,z)) - \frac{\partial_{ss}^2 u_*(t,m_u(t,z))}{|\partial_s u_*(t,m_u(t,z))|^3}\kappa(h) + O(h^{3/2})\right).$$

Then, the result follows from (40) substituting z by $u_*(t,s)$. $\square$

6 Conclusions

In this paper we studied a general class of nonlinear integro-differential operators with important imaging applications, such as the denoising-segmentation Neighborhood filtering.

Although the corresponding PDE problem is multi-dimensional, we showed that it can be reformulated as a one-dimensional problem by means of the notion and properties of the decreasing rearrangement function. We proved the well-posedness of the problem and some stability properties of the solution, as well as the equivalence between the multi-dimensional and the one-dimensional solutions to the problem.

Some other interesting properties were deduced for the rearranged one-dimensional version of the problem, such as the time invariance of the level sets of the solution (inherited by the multidimensional equivalent solution), and the asymptotic behavior of the solution as a shock-type filter.

Future work will point to the use of rearranging techniques for the generalization of the model to include nonlocal effects induced by non-homogeneous spatial kernels, like in equation (4). As already showed for the discrete time problem [13], this situation is much more involved suggesting the consideration of the *relative rearrangement* functional.

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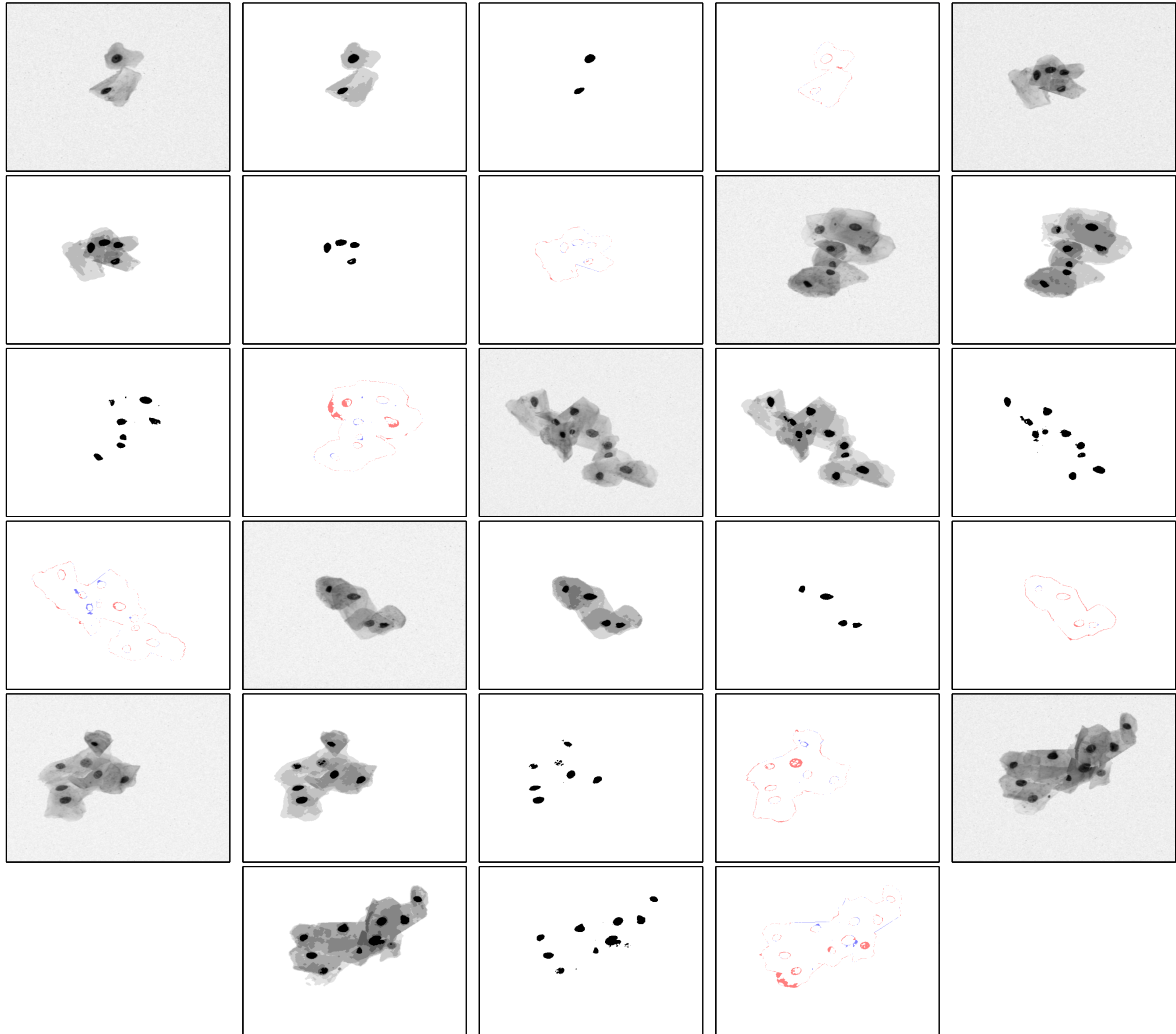[32] L.P. Yaroslavsky, M. Eden, Fundamentals of Digital Optics, Birkhäuser, Boston, 2003.

Figure 1: Example. Nucleus and citoplasm segmentation process. First column corresponds to the initial image. Second column, to the background extraction, and third column to the nucleus segmentation. The citoplasm is the difference between the images shown in the third and second columns. Finally, fourth column shows the difference between the ground-truth nucleus segmentation and the obtained with our method.